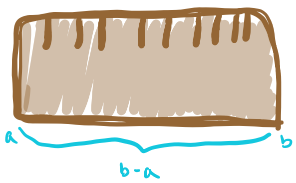
Volumes

We have been finding the area of 2 dimensional regions but now lets learn how we can find the volume of a 3 dimensional region!

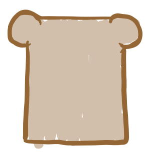
$$\text{Volume} = \int_a^b A(x)\,dx$$

where $A(x)$ is a cross-section of the shape. Lets think about a loaf of bread:



(side view of entire loaf)

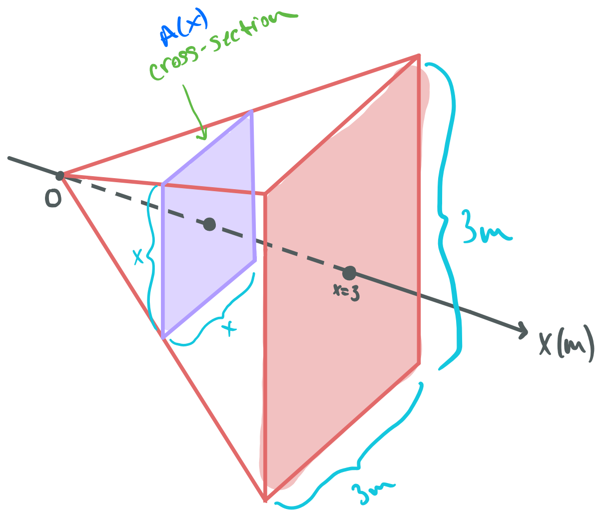
$A(x)$ would be a 2-dimensional slice with $b-a$ = length of the loaf



(front view of $A(x)$ cross-section)

Square Cross Section

ex) A pyramid 3m high has congruent triangular sides and a square base that is 3m on each side. Each cross-section of the pyramid parallel to the base is a square. The height of each square is equal to its current x value. Find volume!



Step 1: Find $A(x)$

cross-section is a square with area $= x^2$

$$\Rightarrow A(x) = x^2$$

Step 2: Find bounds a, b

$a = 0$

$b = 3$ because it's "3m high"

Step 3: Set up integral & Integrate

$$\int_0^3 x^2\,dx = \Big[\tfrac{1}{3}x^3\Big]_0^3 = \boxed{9m^3}$$

Circular Cross Section - Disk Method

Only thing that changes is $A(x)$. To understand revolutions, think about rotating a 2-D circle 360° around its center, this will create a sphere.

ex) The region R between $f(x) = 2 + x \cdot \cos(x)$ and the x-axis from $[-2,2]$ is rotated about the x-axis to create a solid shape. Find Volume:

ⓐ Cross-Section is a circle with area = πr^2 where $r = f(x)$

$$A(x) = \pi(f(x))^2 = \pi(2 + x \cdot \cos(x))^2$$

ⓑ Bounds are given

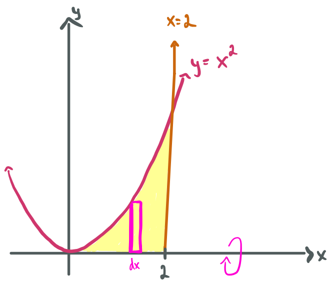
$a = -2$

$b = 2$

ⓒ Set up & Integrate

$$\int_{-2}^{2} \pi(2 + x \cdot \cos(x))^2 \, dx = \pi \int_{-2}^{2} (2 + x \cdot \cos(x))^2 \, dx = \boxed{52.43 \text{ units}^3}$$

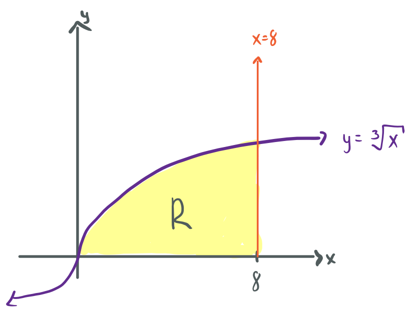
ex| Find the volume of the solid generated by revolving the region bounded by the graphs $y = x^2$, $x = 0$, and $x = 2$ about the x-axis.



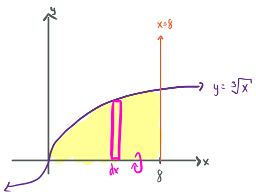
$$A(x) = \pi (x^2)^2$$

$$\pi \int_0^2 (x^2)^2 \, dx = \left[\frac{\pi}{5} x^5\right]_0^2 = \boxed{\frac{32}{5}\pi}$$

ex| Region R is in the first quadrant bounded by the graphs $y = \sqrt[3]{x}$ and $x = 8$

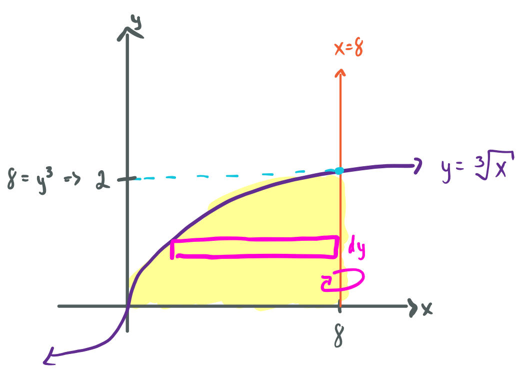


(a) Find the volume of the solid generated by rotating R around the x-axis



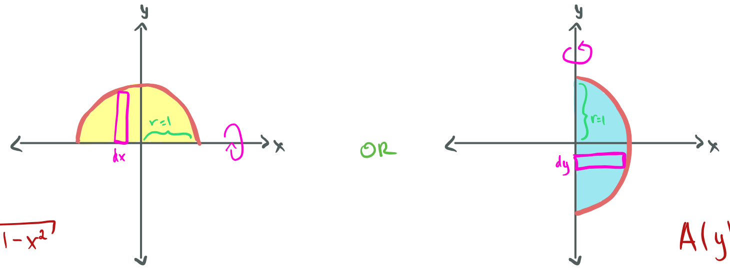
$$\pi \int_0^8 \left(\sqrt[3]{x}\right)^2 dx = \pi \left[\frac{3}{5} x^{5/3}\right]_0^8 \approx \boxed{19.2\pi}$$

(b) Find the volume of the solid generated by rotating R around the line x = 8



$$\pi \int_0^2 (8 - y^3)^2 \, dy = \pi \int_0^2 64 - 16y^3 + y^6 \, dy = \pi \left[64y - 4y^4 + \frac{1}{7} y^7\right]_0^2 \approx \boxed{82\pi}$$

ex | Find the Volume of a sphere with a radius of 1.



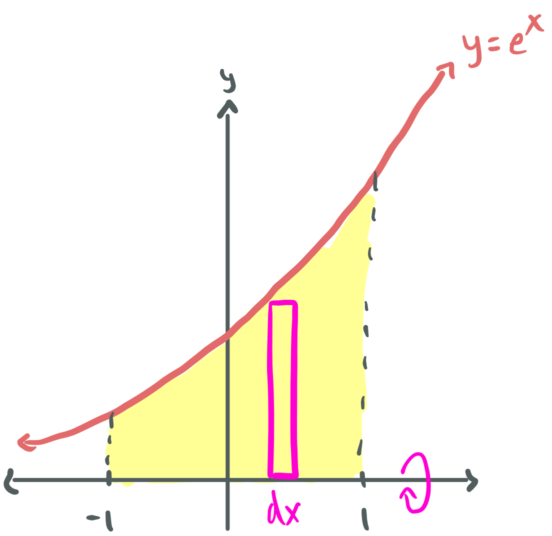
$A(x) = \sqrt{1-x^2}$ $\qquad$ $A(y) = \sqrt{1-y^2}$

With Respect to x: $\pi \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^2 dx = \pi \left[x - \frac{1}{3}x^3 \right]_{-1}^{1} = \pi\left(\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) \right) = \frac{4}{3}\pi$

With Respect to y: $\pi \int_{-1}^{1} \left(\sqrt{1-y^2}\right)^2 dy$ OR $2\cdot\pi \int_{0}^{1} \left(\sqrt{1-y^2}\right)^2 dy = 2\pi \left[y - \frac{1}{3}y^3 \right]_{0}^{1} = 2\pi\left(\left(1-\frac{1}{3}\right) \right) = \frac{4}{3}\pi$

Using Geometry: $A_{sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$

Ex | Find the volume of the solid generated by revolving the region bounded by the graphs $y = e^x$, $y = 0$, $x = -1$, $x = 1$ about the x-axis.



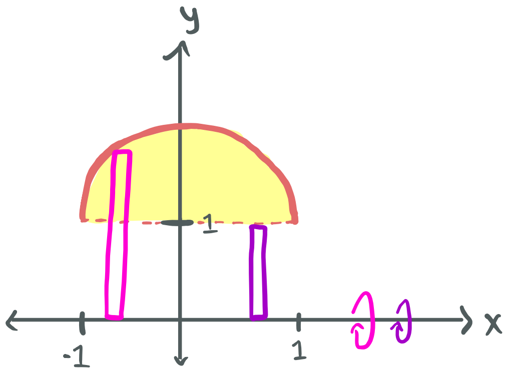
$A(x) = \pi(e^x)^2 = \pi e^{2x}$

$\pi \int_{-1}^{1} e^{2x}\, dx = 11.394$ units^3

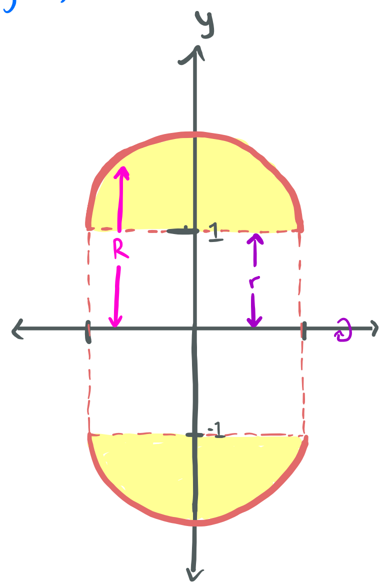
Washer Method

(Disk method with a gap between the Region and axis of rotation)

ex Region R is defined by a horizontally-oriented semi-circle with radius $r=1$ that is shifted up 1 unit from the x-axis. Find the volume generated when Region R is rotated about the x-axis. Set up the integral, don't solve.



Revolve →



✱ Like a wedding ring

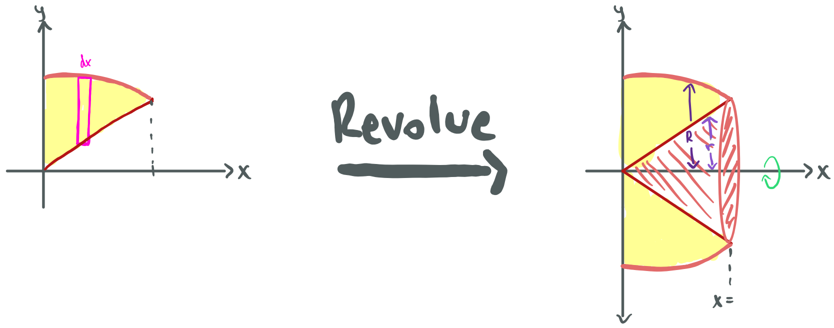
$$A(x) = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi\left(\left(\sqrt{1-x^2}+1\right)^2 - (1)^2\right)$$

$$= \pi\left(1-x^2 + 2\sqrt{1-x^2} + 1 - 1\right)$$

$$= \pi\left(1 - x^2 + 2\sqrt{1-x^2}\right)$$

$$\pi\int_{-1}^{1} 1 - x^2 + 2\sqrt{1-x^2}\,dx = \boxed{4.47\pi}$$

ex] The region R is enclosed by the y-xis and the graphs of $y = \cos(x)$ and $y = \sin(x)$ is revolved about the x-axis to form a solid. Find volume.



☆ Hollow inside

ⓐ Find $A(x)$

It's revolving so we know $A(x) = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

$$= \pi\left(\cos^2(x) - \sin^2(x)\right)$$

ⓑ Find bounds

$a = 0$ given since bounded by y-axis

b: where $\cos(x) = \sin(x)$ Thus, $b = \frac{\sqrt{2}}{2} = \frac{\pi}{4}$

ⓒ Set up & Integrate

$$\int_0^{\frac{\pi}{4}} \pi\left(\cos^2(x) - \sin^2(x)\right)\,dx = \int_0^{\frac{\pi}{4}} \pi(\cos(x))^2\,dx - \int_0^{\frac{\pi}{4}} \pi(\sin(x))^2\,dx = \boxed{\frac{\pi}{2}\ \text{units}^3}$$

We're just subtracting the sin cone from the larger cos outter shell

Trig Identity

$$\boxed{☆\ \cos^2(x) - \sin^2(x) = \cos(2x)}$$

Ex | Find the volume of the solid generated by revolving the region bounded by the graphs $y = x^2+1$, $y = x+3$ about the x-axis.

Bounds: $x^2+1 = x+3$

$x^2 - x - 2 = 0$

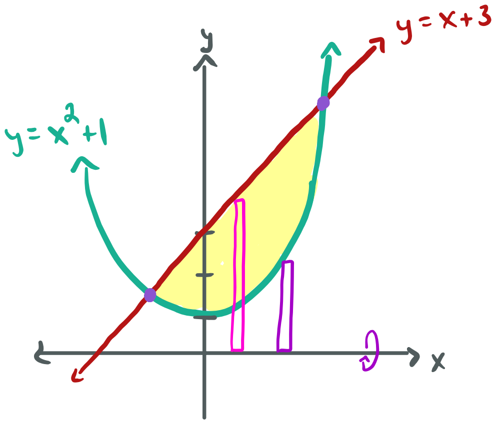
$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$A(x) = \pi(R^2 - r^2) = \pi\left((x+3)^2 - (x^2+1)^2\right)$

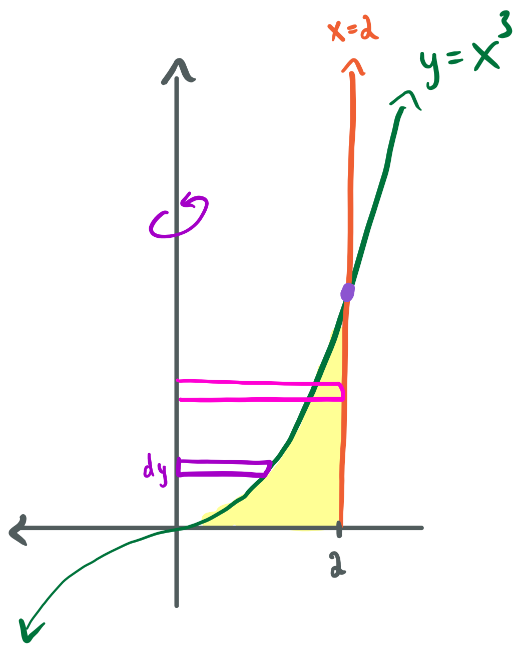
$= \pi\left(x^2 + 6x + 9 - (x^4 + 2x^2 + 1)\right)$

$= \pi\left(-x^4 - x^2 + 6x + 8\right)$



$$\pi\int_{-1}^{2} (-x^4 - x^2 + 6x + 8)\,dx = \boxed{23.4\pi}$$

Ex | Find the volume of the solid generated by revolving the region bounded by $y = x^3$, $y = 0$, $x = 2$ about the y-axis.



$A(x) = \pi(R^2 - r^2) = \pi\left((2)^2 - (\sqrt[3]{y})^2\right)$

$= \pi(4 - y^{2/3})$

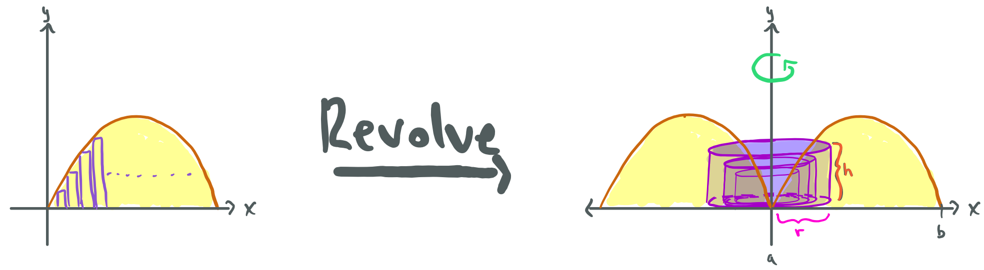
Bounds: $\sqrt[3]{y} = 2$

$y = 8,\ y = 0$

$$\pi\int_{0}^{8} 4 - y^{2/3}\,dy = \boxed{12.8\pi}$$

Shell Method — Ring Cross Sections

Instead of revolving a 2D circle 360° around an axis in space, we can sum up Rings that keep expanding in diameter.



$A(x) = 2\pi r h = 2\pi x \cdot f(x)$ = Surface Area of a ring

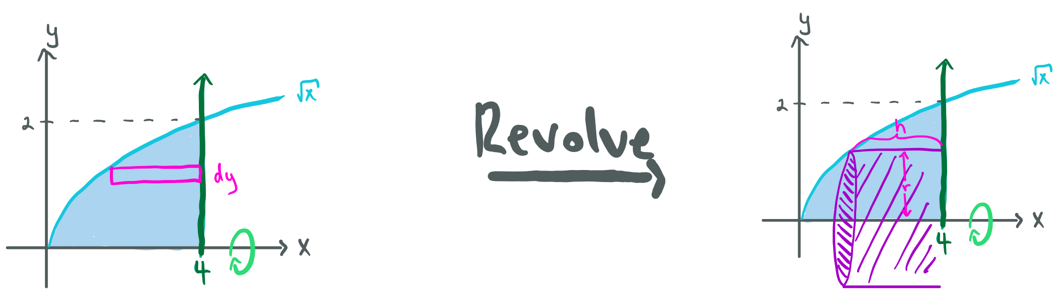
$2\pi x \cdot f(x) \cdot dx$ = Volume of a ring

$$\boxed{R = 2\pi \int_a^b x \cdot f(x)\, dx}$$ = Sum of volume of all rings

☆ Ring's sides Must! be parallel to axis of rotation

https://www.youtube.com/watch?v=NyBX5DIcAMg

ex The region R is bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find volume.



ⓐ Rewrite in terms of y

$y=\sqrt{x} \Rightarrow x=y^2$

ⓑ Find bounds

$4=y^2 \Rightarrow b=2, \quad a=0$

✯ Must be in terms of y to use Shell method because we're revolving around x-axis

ⓒ Find height & radius of cylindrical cross-section

$h=4-y^2$

$r=y$

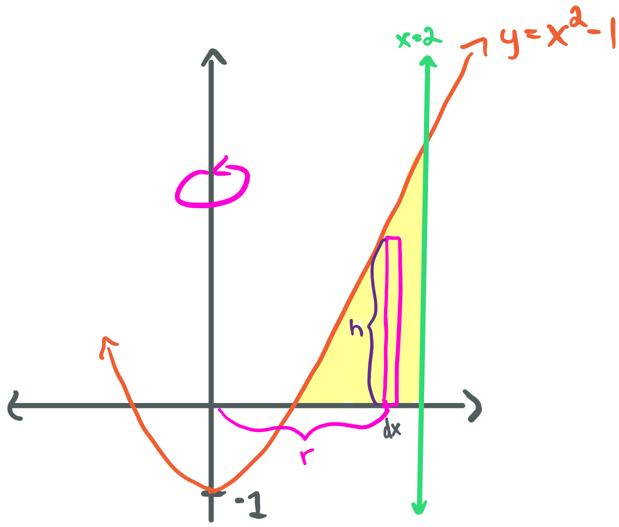
ⓓ Find Area of each cross-section A(x)

$A(x)=2\pi rh=2\pi y(4-y^2)=2\pi(4y-y^3)$

ⓔ Set up integral & Integrate

$$R=2\pi\int_0^2 \left[4y-y^3\right]dy = 2\pi\left[2y^2-\frac{1}{4}y^4\right]_0^2 = 2\pi\left(2(2)^2-\frac{1}{4}(2)^4\right) = \boxed{8\pi \text{ units}^3}$$

ex) Find the volume of the solid generated by revolving the region bounded by graphs $y=x^2-1$, $x=2$, $y=0$ around y-axis.

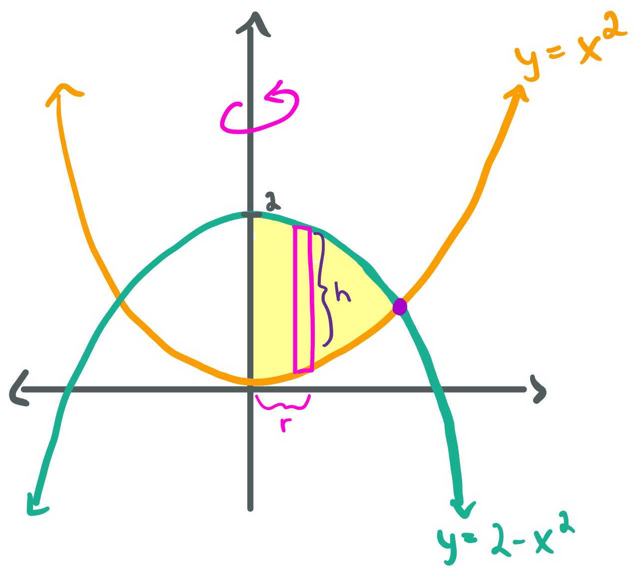


$$A(x) = 2\pi r h = 2\pi x\left(x^2-1\right)$$

Bounds : Radii of cylinders are from $x=2$ to $0=x^2-1 \Rightarrow x=1$

$$2\pi\int_1^2 x\left(x^2-1\right)dx = \boxed{4.5\pi}$$

ex) Find the volume of the solid generated by revolving the region bounded by the graphs $y = 2 - x^2$, $y = x^2$, $x = 0$ around the y-axis.



$$A(x) = 2\pi r h = 2\pi x\left(2 - x^2 - x^2\right)$$

$$= 4\pi x\left(1 - x^2\right)$$

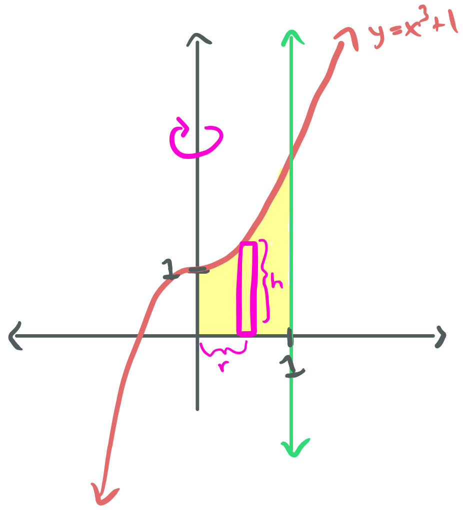
Bounds: $x^2 = 2 - x^2$

$x^2 = 1$

Radii from $x = 0$ to $x = 1$

$$4\pi \int_0^1 x\left(1 - x^2\right)\,dx = \boxed{\pi}$$

ex| Revolve $y = x^3 + 1$, $x = 1$, $x = y = 0$ about the y-axis



Shell:

$A(x) = 2\pi r h = 2\pi x(x^3+1)$

Bounds: $0 \le x \le 1$

$\Rightarrow 2\pi \int_0^1 x(x^3+1)\,dx = \boxed{4.39 \text{ units}^3}$

Washer:

$y = x^3+1 \Rightarrow x = \sqrt[3]{y-1}$

$A_1(y) = \pi r^2 \qquad A_2(y) = \pi(R^2 - r^2)$

Bounds: Need to split: $y=0 \rightarrow y=1$

$\sqrt[3]{y-1} = 1$ @ $y = 2 \Rightarrow y=1 \rightarrow y=2$

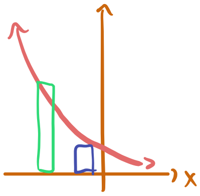
$y=0 \rightarrow y=1$: $A_1(y) = \pi(1)^2 = \pi$

$y=1 \rightarrow y=2$: $A_2(y) = \pi\left(1 - (y-1)^{2/3}\right)$

$\pi \int_0^1 dy + \pi \int_1^2 \left[1 - (y-1)^{2/3}\right] dy = \boxed{4.39 \text{ units}^3}$

AP Calculus BC Assignment: 7.1-7.2

1. If $A = \int_0^1 e^{-x} dx$ is approximated using Riemann sums and the same number of subdivisions, and if L, R, and T denote, respectively left, right, and trapezoidal sums, then it follows that



(a) $R \leq A \leq T \leq L$ (b) $R \leq T \leq A \leq L$ (c) $L \leq T \leq A \leq R$

(d) $L \leq A \leq t \leq R$ (e) None of these is true.

2. If $\dfrac{dy}{dx} = y tan x$ and $y = 3$ when $x = 0$, then, when $x = \frac{\pi}{3}$, $y =$

$\int \frac{1}{y} dy = \int \tan(x)\, dx \Rightarrow \ln(y) = -\ln(\cos(x)) + C \Rightarrow y = Ce^{-\ln(\cos(x))} \Rightarrow y = \frac{C}{\cos(x)}$ $3 = C \Rightarrow y = \frac{3}{\cos(x)}$ $y(\frac{\pi}{3}) = \frac{3}{(\frac{1}{2})} = 6$

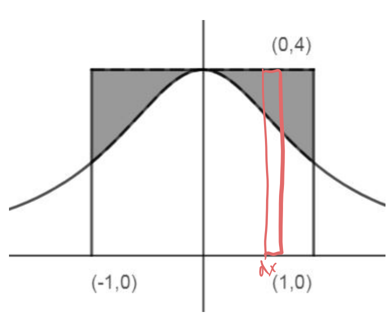
(a) $ln\sqrt{3}$ (b) $ln3$ (c) $\frac{3}{2}$ (d) $\frac{3\sqrt{3}}{2}$ (e) 6

3. $\displaystyle\int_0^6 f(x-1)\, dx =$ $F(5) - F(-1)$

(a) $\displaystyle\int_{-1}^7 f(x)\, dx$ (b) $\displaystyle\int_{-1}^5 f(x)\, dx$ (c) $\displaystyle\int_{-1}^5 f(x+1)\, dx$

(d) $\displaystyle\int_1^5 f(x)\, dx$ (e) $\displaystyle\int_1^7 f(x)\, dx$

4. The equation of the curve shown below is $y = \frac{4}{1+x^2}$. What does the area of the shaded region equal?



$A(x) = 4 - \frac{4}{1+x^2}$

$2\int_0^1 4 - \frac{4}{1+x^2}\, dx = 1.71$

(a) $4 - \frac{\pi}{4}$ (b) $8 - 2\pi$ (c) $8 - \pi$ (d) $8 - \frac{\pi}{2}$ (e) $2\pi - 4$

$\frac{dP}{dt} = kP \Rightarrow P(t) = P_0 e^{kt}$ $\quad$ $\frac{4}{5}P_0 = P_0 e^{k}$ $\quad$ $k = \ln\left(\frac{4}{5}\right)$ $\quad$ $\Rightarrow P(t) = P_0 e^{t\ln\left(\frac{4}{5}\right)}$ $\quad$ $0.02P_0 = P_0 e^{t\ln\left(\frac{4}{5}\right)}$ $\quad$ $\Rightarrow t = \frac{\ln(0.02)}{\ln\left(\frac{4}{5}\right)} = 17.53$ mins

5. The rate at which a purification process can remove contaminants from a tank of water is proportional to the amount of contaminant remaining. If 20% of the contaminant can be removed during the first minute of the process and 98% must be removed to make the water safe, approximately how long will the decontamination process take? $\quad P(1) = \frac{4}{5}P_0 \qquad P(t) = 0.02P_0$

(a) $2min$ $\qquad$ (b) $5min$ $\qquad$ (c) $18min$ $\qquad$ (d) $20min$ $\qquad$ (e) $40min$

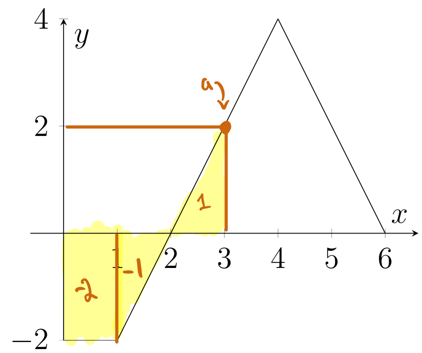
6. Let $H(x) = \int_0^x f(t)\,dt$, where f is the function whose graph appears below

$H(x) = F(x) - \cancel{F(0)}^{\,0}$

$H(3) = F(3) = -2$

$H'(x) = f(x)$

$H'(3) = f(3) = 2$



$y = H(a) + H'(a)(x-a)$

$y = H(3) + H'(3)(x-3)$

$= -2 + 2(x-3)$

$= -2 + 2x - 6$

$= -8 + 2x$

The local linearization of $H(x)$ near $x = 3$ is $H(x) =$

(a) $-2x+8$ $\qquad$ (b) $2x-4$ $\qquad$ (c) $-2x+4$ $\qquad$ (d) $2x-8$ $\qquad$ (e) $2x-2$

7. The table shows the speed of an object, in feet per second, during a 3-second period.

$time\,(sec)$	0	1	2	3
$speed\,(ft/sec)$	30	22	12	0

Estimate the distance the object travels, using the trapezoidal method.

(a) $34ft$ $\qquad$ (b) $45ft$ $\qquad$ (c) $48ft$ $\qquad$ (d) $49ft$ $\qquad$ (e) $64ft$

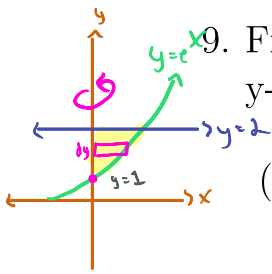
$\frac{22+30}{2} + \frac{12+22}{2} + \frac{0+12}{2} = 26 + 17 + 6 = 49ft$

$\frac{1}{10}\int_0^{10} 70+50e^{-0.4t}\,dt = 82.3°$

$120 = 70+k \qquad k = 50$

8. As a cup of hot chocolate cools, its temperature after t minutes is given by $H(t) = 70 + ke^{-0.4t}$. If its initial temperature was $120°F$, what was its average temperature ($in°F$) during the first 10 minutes?

   (a) 60.9 (b) 82.3 (c) 95.5 (d) 96.1 (e) 99.5



9. Find the volume of the solid generated when the region bounded by the y-axis, $y = e^x$, and $y = 2$ is rotated around the y-axis. $\pi\int_1^2 (\ln(y))^2\,dy$

   (a) 0.296 (b) 0.592 (c) 2.427 (d) 3.998 (e) 27.577

10. If $f(t) = \int_0^{t^2} \frac{1}{1+x^2}dx$ then $f'(t) =$ $f(t) = F(t^2) - F(0)$ $\quad f'(t) = f(t^2)\cdot 2t = \frac{2t}{1+t^4}$

    (a) $\frac{1}{1+t^2}$ (b) $\frac{2t}{1+t^2}$ (c) $\frac{1}{1+t^4}$ (d) $\frac{2t}{1+t^4}$ (e) $tan^{-1}t^2$

11. At how many points on the interval $[0, \pi]$ does $f(x) = 2sinx + sin4x$ satisfy the Mean Value Theorem? $\exists c \in [0,\pi]$ s.t. $f'(c) = \frac{f(\pi)-f(0)}{\pi} = 0$ $\quad f'(x) = 2\cos(x) + 4\cos(4x) = 0$

    (a) none (b) 1 (c) 2 (d) 3 (e) 4

12. Let $f(x) \begin{cases} 1+e^{-x}, & 0 \le x \le 5 \\ 1+e^{x-10}, & 5 \le x \le 10 \end{cases}$ $\quad$ @ $x=5 \Rightarrow 1+e^{-5} = 1+e^{5-10}$ ✓

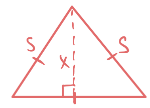
    Which of the following statements are true?

    ✓ I. $f(x)$ is continuous for all values of x in the interval $[0, 10]$.

    II. $f'(x)$ the derivative of $f(x)$, is continuous for all values of x in the interval $[0, 10]$. $\lim_{x\to 5^-} \neq \lim_{x\to 5^+}$

    ✓ III. The graph of $f(x)$ is concave upwards for all values of x in the interval $[0, 10]$

    (a) I only (b) II only (c) III only
    (d) I and II only (e) I, II, and III



$X = \frac{\sqrt{3}}{2} s$

13. A solid has a circular base of radius 3. If every plane cross section perpendicular to the $x - axis$ is an equilateral triangle, then its volume is

(a) 36 (b) $12\sqrt{3}$ (c) $18\sqrt{3}$ (d) $24\sqrt{3}$ (e) $36\sqrt{3}$

14. The base of a solid is the region in the first quadrant bounded by the line $x + 2y = 4$ and the coordinate axes. What is the volume of the solid if every cross section perpendicular to the $x - axis$ is a semicircle? $y = \frac{4-x}{2}$

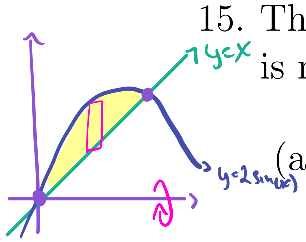
$\frac{1}{2}\pi r^2$

$\frac{\pi}{2}\int_0^4 \left(\frac{1}{2}\cdot\frac{4-x}{2}\right)^2 dx =$

(a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{8\pi}{3}$ (d) $\frac{32\pi}{3}$ (e) $\frac{64\pi}{3}$

$\star\ r = \frac{1}{2}y$

15. The region in the first quadrant enclosed by the graphs $y = x$ and $y = 2sinx$ is revolved about the x-axis. The volume of the solid generated is $2\sin(x) = x$ @ $x = 1.895$



(a) 1.895 (b) 2.126 (c) 5.811 (d) 6.678 (e) 13.355

$\pi \int_0^{1.895} (2\sin(x))^2 - (x)^2\, dx$

16. If the length of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by $L = \int_a^b \sqrt{e^{2x} + 2e^x + 2}dx$ then $f(x) =$

$1 + (f'(x))^2 = e^{2x} + 2e^x + 2$

$f'(x) = \sqrt{e^{2x} + 2e^x + 1} = \sqrt{(e^x+1)(e^x+1)}$

$f'(x) = e^x + 1$

$f(x) = e^x + x + C$

(a) $2e^{2x} + 2e^x$ (b) $\frac{1}{2}e^{2x} + 2e^x + 2x$ (c) $e^x - x + 3$

(d) $e^x + 1$ (e) $e^x + x - 2$

17. The length of the curve $y = x^3$ from $(0, 0)$ to $(1, 1)$ $\int_0^1 \sqrt{1 + (3x^2)^2}\, dx$

(a) 1.380 (b) 1.414 (c) 1.495 (d) 1.548 (e) 1.732

18. If $f(x) > 0$ is continuous and $g(x) = \int_0^x \sqrt{(f(t))^2 - 1}dt$, what is the length of the graph of $g(x)$ from $x = a$ to $x = b$?

$g'(x) = \sqrt{(f(x))^2 - 1}$ $(g'(x))^2 = [f(x)]^2 - 1$

(a) $\int_a^b f(x)\, dx$ (b) $\int_a^b g(x)\, dx$ (c) $\int_a^b \sqrt{(f(x))^2 + 1}dx$

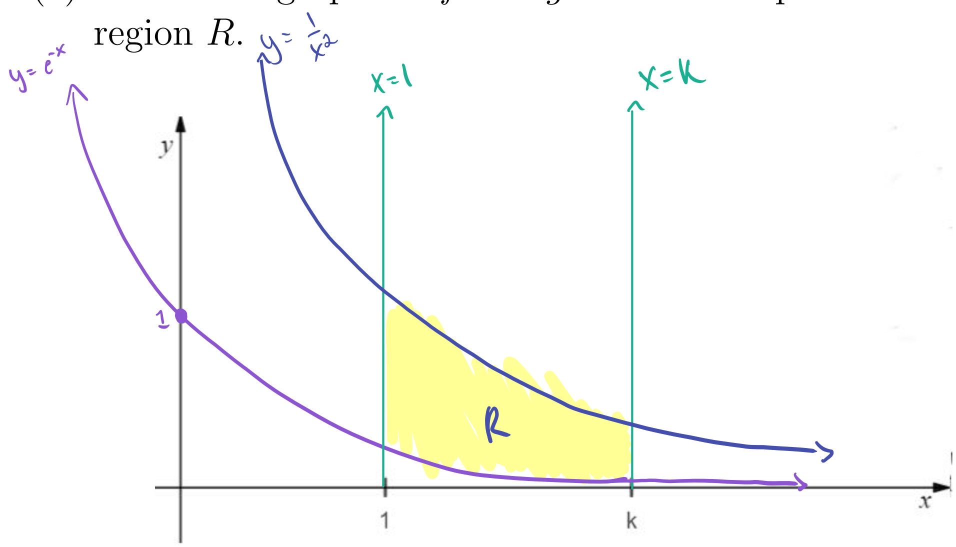
(d) $\int_a^b \sqrt{g(x) + 1}dx$ (e) $\int_a^b \sqrt{(g(x))^2 + 1}dx$

$\int_a^b \sqrt{1 + [f(x)]^2 - 1}\, dx$

$\int_a \sqrt{1+[g'(x)]^2}\, dx$

19. Let R be the region enclosed by the graphs of $f(x) = \frac{1}{x^2}, g(x) = e^{-x}$, and the lines $x = 1$ and $x = k$ where $k > 1$.

   (a) Sketch the graphs of f and g on the axes provided below and shade the region R.



   (b) Without using absolute value, set up and evaluate in terms of k, an integral expression that gives $A(k)$, the area of region R.

   (c) Find $\lim_{k \to \infty} A(k)$.

   (d) Let $k = 4$. Find the volume of the solid generated when region R is revolved around the $x - axis$.

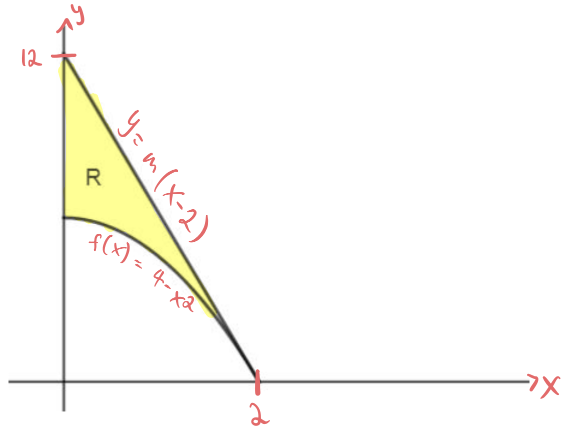
(b) $A(k) = \int_1^k \frac{1}{x^2} - e^{-x}\, dx = \left[-\frac{1}{x} + e^{-x}\right]_1^k = \left(-\frac{1}{k} - e^{-k}\right) - \left(-1 + \frac{1}{e}\right)$

$\boxed{= -\frac{1}{k} - e^{-k} + 1 - \frac{1}{e}}$

(c) $\lim_{k \to \infty} \left[\cancel{-\frac{1}{k}}^{\,0} - \cancel{e^{-k}}^{\,0} - 1 + \frac{1}{e}\right] = \boxed{1 - \frac{1}{e}}$

(d) $\pi \int_1^4 \left[\frac{1}{x^4} - \frac{1}{e^{2x}}\right] dx = \boxed{0.81 \text{ units}^3}$

20. As shown in the diagram below, the region R lies in the first quadrant above the graph of $f(x) = 4 - x^2$ and below the line $y = m(x-2)$.



$-2m = 12$

$m = -6$

$y = -6x + 12$

(a) If in the first quadrant the line lies above the graph of f, determine the range of m.

(b) When the line intersects the $y-axis$ at $(0,12)$, what is the ~~region~~ area of R?

(c) Write an expression without an integral sign for A(m), the area of R in terms of m

(d) If m is changing at the constant rate of -2 units per second, how fast is $A(m)$ changing at the instant the line intersects the axis at $(0,12)$? Is the area increasing or decreasing?

(a) $f'(x) = -2x$ @ $x=2$: $f'(2) = -4$ $\boxed{m \leq -4}$

(b) $\int_0^2 (-6x+12) - (4-x^2)\,dx = \int_0^2 x^2 - 6x + 8\,dx = \boxed{\frac{20}{3} \text{ units}^2}$

(c) $\frac{1}{2}(2)(-2m) = -2m$

$\int_0^2 4 - x^2\,dx = \frac{16}{3}$

$\boxed{R = A(m) = -2m - \frac{16}{3}}$

(d) Given: $\frac{dm}{dt} = -2$

$$\frac{dA}{dt} = -2\frac{dm}{dt} = -2(-2) = \boxed{+4 \text{ Inc}}$$